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# Proof of Kochen–Specker Theorem: Conversion of Product Rule to Sum Rule *

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*Valuation functions of observables in quantum mechanics are often expected to obey two constraints called the sum rule and product rule. However, the Kochen–Specker (KS) theorem shows that for a Hilbert space of quantum mechanics of dimension $d \geq 3$, these constraints contradict individually with the assumption of value definiteness. The two rules are not irrelated and Peres [Found. Phys. 26 (1996) 807] has conceived a method of converting the product rule into a sum rule for the case of two qubits. Here we apply this method to a proof provided by Mermin based on the product rule for a three-qubit system involving nine operators. We provide the conversion of this proof to one based on sum rule involving ten operators.*

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Quantum mechanical states impose statistical restrictions on the results of measurements. Some physicists think that the restriction is due to the incomplete description of the quantum system by the states and hence they propose hidden variable theories in the hope that these would give a complete description. The Kochen–Specker (KS) theorem provides a strong argument showing that even if hidden variables do exist and can be used to interpret quantum mechanics, the value assignments made must be contextual. A value of an observable is said to be contextual if the value measured depends on measurement context. Apart from noncontextuality, value definiteness is another old belief held by physicists which states that all (compatible) observables of a physical system have definite values at all times. Unfortunately, applying noncontextuality and value definiteness to a quantum mechanical system would give rise to a contradiction.[1-3]

In quantum mechanics, observables are represented by Hermitian operators that have real eigenvalues. Quantum mechanics requires that the results of measuring an observable be eigenvalues of the corresponding Hermitian operator. Quantum mechanics further requires that if observables $A, B, C, \ldots$ belong to mutually commuting subsets of the observables and satisfy $f(A, B, C, \ldots) = 0$, then the only allowed results of a simultaneous measurement of $A, B, C, \ldots$ are the set of simultaneous eigenvalues $v(A)$, $v(B)$, $v(C), \ldots$ constrained by $f(v(A), v(B), v(C), \ldots) = 0$. In particular, this can be expressed as a sum rule: (a) If $A, B, C$ are the compatible observables and $C = A + B$, then $v(C) = v(A) + v(B)$. Alternatively, one can have the product rule, i.e. (b) if $A, B, C$ are the compatible observables and $C = A \cdot B$, then $v(C) = v(A) \cdot v(B)$.[1]

The KS theorem states that for Hilbert space of quantum mechanical state vectors of dimension $> 2$, assumptions of value definiteness contradict either the sum rule or product rule. To avoid contradiction and to maintain the value definiteness, the measured values of observables must be contextual.

Consider a pair of spin-1/2 particles and their spin observables. The following magic square in Table 1 consists of nine tensor-product spin operators given by Mermin.[3] As usual, $\sigma_x$, $\sigma_y$, $\sigma_z$ are the Pauli matrices related to the individual spin observables.

Table 1. Mermin's magic square.

| | | |
|---|---|---|
| $I \otimes \sigma_z$ | $\sigma_z \otimes I$ | $\sigma_z \otimes \sigma_z$ |
| $\sigma_x \otimes I$ | $I \otimes \sigma_x$ | $\sigma_x \otimes \sigma_x$ |
| $\sigma_x \otimes \sigma_z$ | $\sigma_z \otimes \sigma_x$ | $\sigma_y \otimes \sigma_y$ |

Each row and each column is a triad of commuting operators. Each entry of the operator has eigenvalue 1 or $-1$. We can easily check the validity of following six functions,

$$\begin{aligned}(I \otimes \sigma_z)(\sigma_z \otimes I)(\sigma_z \otimes \sigma_z) &= I \otimes I,\\ (\sigma_x \otimes I)(I \otimes \sigma_x)(\sigma_x \otimes \sigma_x) &= I \otimes I,\\ (\sigma_x \otimes \sigma_z)(\sigma_z \otimes \sigma_x)(\sigma_y \otimes \sigma_y) &= I \otimes I,\\ (I \otimes \sigma_z)(\sigma_x \otimes I)(\sigma_x \otimes \sigma_z) &= I \otimes I,\\ (\sigma_z \otimes I)(I \otimes \sigma_x)(\sigma_z \otimes \sigma_x) &= I \otimes I,\\ (\sigma_z \otimes \sigma_z)(\sigma_x \otimes \sigma_x)(\sigma_y \otimes \sigma_y) &= -I \otimes I. \qquad (1)\end{aligned}$$

According to the product rule, we have

$$\begin{aligned}v(I \otimes \sigma_z)v(\sigma_z \otimes I)v(\sigma_z \otimes \sigma_z) &= v(I \otimes I),\\ v(\sigma_x \otimes I)v(I \otimes \sigma_x)v(\sigma_x \otimes \sigma_x) &= v(I \otimes I),\\ v(\sigma_x \otimes \sigma_z)v(\sigma_z \otimes \sigma_x)v(\sigma_y \otimes \sigma_y) &= v(I \otimes I),\\ v(I \otimes \sigma_z)v(\sigma_x \otimes I)v(\sigma_x \otimes \sigma_z) &= v(I \otimes I),\end{aligned}$$

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